

Maximum principle for optimal control of stochastic partial differential equations*

AbdulRahman Al-Hussein

*Department of Mathematics, College of Science, Qassim University,
P.O.Box 6644, Buraydah 51452, Saudi Arabia*

E-mail: alhusseinqu@hotmail.com

Abstract

We shall consider a stochastic maximum principle of optimal control for a control problem associated with a stochastic partial differential equations of the following type:

$$\begin{cases} dx(t) = (A(t)x(t) + a(t, u(t))x(t) + b(t, u(t)))dt \\ \quad + [\langle \sigma(t, u(t)), x(t) \rangle_K + g(t, u(t))]dM(t), \\ x(0) = x_0 \in K, \end{cases}$$

with some given predictable mappings a, b, σ, g and a continuous martingale M taking its values in a Hilbert space K , while $u(\cdot)$ represents a control. The equation is also driven by a random unbounded linear operator $A(t, w)$, $t \in [0, T]$, on K .

We shall derive necessary conditions of optimality for this control problem without a convexity assumption on the control domain, where $u(\cdot)$ lives, and also when this control variable is allowed to enter in the martingale part of the equation.

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1 Introduction

Consider the following stochastic partial differential equation (SPDE for short):

$$\begin{cases} dx(t) = (A(t)x(t) + a(t, u(t))x(t) + b(t, u(t)))dt \\ \quad + [\langle \sigma(t, u(t)), x(t) \rangle_K + g(t, u(t))]dM(t), \quad 0 \leq t \leq T, \\ x(0) = x_0 \in K, \end{cases} \quad (1.1)$$

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where $A(t), t \in [0, T]$, is a random unbounded closed linear operator on a separable Hilbert space K . The noise is modelled by a continuous martingale M in K and a, b, σ and g are suitable predictable bounded mappings while $u(\cdot)$ is a control. This equation will be studied over a Gelfand triple (V, K, V') . That is V is a separable Hilbert space embedded continuously and densely in K . More precisely, given a bounded measurable mapping $\ell : [0, T] \times \mathcal{O} \rightarrow K$ and a fixed element G of K , we shall be interested here in minimizing the cost functional:

$$J(u(\cdot)) = \mathbb{E} \left[\int_0^T \langle \ell(t, u(t)), x^{u(\cdot)}(t) \rangle_K dt + \langle G, x^{u(\cdot)}(T) \rangle_K \right],$$

over the set of admissible controls. We will approach this by using the adjoint equation of the SPDE (1.1), which is a backward stochastic partial differential equation (BSPDE) driven by an infinite dimensional martingale, and derive in particular a stochastic maximum principle for this optimal control problem. Such BSPDEs (or even BSDEs) have their importance shown in applications in control theory like [5] and in some financial applications as in [20]. For more applications we refer the reader to Bally et al. [8], Imkeller et al. [16] and [12].

It is known that a Wiener filtration is usually required to deal with BSPDEs that arise as adjoint equations of controlled SPDEs. This is indeed a restriction insisted on for example in [30] and [31]. Øksendal et al. in [23] and some other recent works have now considered the adjoint equation of a controlled BSPDE with a filtration generated by a Wiener process and a Poisson random measure. In our work here we can consider an arbitrary continuous filtration thanks to a result established in [3] giving existence and uniqueness of solutions to BSPDEs driven by martingales. In this respect we refer the reader also to Imkeller et al. [16], where a filtration is being taken which is similar to the one used here. The reader can also see [5], [29], [17], [14], [15], [18], [13], [27] and [25] for SDEs and SPDEs with martingale noises. In fact in [5] we derived necessary conditions for optimality of stochastic systems similar to (1.1), but the result there describes the maximum principle only in a local form and requires moreover the convexity of the control domain U . In the present work we shall derive the maximum principle in its global form for our optimal control problem and, in particular, we shall not require the convexity of U . Moreover, our results here generalize those in [31] and [10] and can be applied to the optimal control problem of partial observations with a given general nonlinear cost functional as done particularly in [31, Section 6]. The idea of reducing such a control problem to a control problem for a linear SPDE (Zakai's equation) was discussed also there. This is similar to (1.1).

The main new features here are the driving noise is allowed to be an infinite dimensional martingale (as in Tudor [29] and Al-Hussein [4]), the control domain U need not be convex, and the control variable itself is allowed to enter in the martingale part of the equation as in the SPDE (1.1).

The present paper is organized as follows. In Section 2 we introduce some definitions and notation that will be used throughout the paper. In Section 3 our main stochastic control problem is introduced. Section 4 is devoted to the adjoint equation of the SPDE (1.1) as well as the existence and uniqueness of its solution. Finally, we state and establish the proof of our main result in Section 5.

2 Basic definitions and Notation

We assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ is our complete filtered probability space, such that $\{\mathcal{F}_t\}_{t \geq 0}$ is a continuous filtration, in the sense that every square integrable K -valued martingale with respect to $\{\mathcal{F}_t, 0 \leq t \leq T\}$ has a continuous version. Let \mathcal{P} denote the predictable σ -algebra of subsets of $\Omega \times [0, T]$. A K -valued process is said to be predictable if it is $\mathcal{P}/\mathcal{B}(K)$ measurable. Let $\mathcal{M}_{[0, T]}^{2, c}(K)$ be the space of all square integrable continuous martingales in K . We say that two elements M and N of $\mathcal{M}_{[0, T]}^{2, c}(K)$ are *very strongly orthogonal* (VSO) if $\mathbb{E}[M(\tau) \otimes N(\tau)] = \mathbb{E}[M(0) \otimes N(0)]$, for all $[0, T]$ -valued stopping times τ .

For $M \in \mathcal{M}_{[0, T]}^{2, c}(K)$ let $\langle\langle M \rangle\rangle$ be its *angle process* taking its values in the space $L_1(K)$, where $L_1(K)$ is the space of nuclear operators on K , and satisfying $M \otimes M - \langle\langle M \rangle\rangle \in \mathcal{M}_{[0, T]}^{2, c}(L_1(K))$, and denote by $\langle M \rangle$ the quadratic variation of M . It is known (see [22]) that there exist a predictable process $\tilde{Q}_M(s, \omega)$ in $L_1(K)$ such that $\langle\langle M \rangle\rangle_t = \int_0^t \tilde{Q}_M(s, \omega) d\langle M \rangle_s$.

For (t, ω) if $\tilde{Q}(t, \omega)$ is any symmetric, positive definite nuclear operator on K , we shall denote by $L_{\tilde{Q}(t, \omega)}(K)$ the set of all linear (not necessarily bounded) operators Φ which map $\tilde{Q}^{1/2}(t, \omega)(K)$ into K such that $\Phi \tilde{Q}^{1/2}(t, \omega) \in L_2(K)$, the space of all Hilbert-Schmidt operators from K into itself. The inner product and norm in $L_2(K)$ will be denoted respectively by $\langle \cdot, \cdot \rangle_2$ and $\|\cdot\|_2$.

We recall that the stochastic integral $\int_0^\cdot \Phi(s) dM(s)$ is defined for mappings Φ such that for each (t, ω) , $\Phi(t, \omega) \in L_{\tilde{Q}_M(t, \omega)}(K)$, for every $h \in K$ the K -valued process $\Phi \tilde{Q}_M^{1/2}(h)$ is predictable, and $\mathbb{E}[\int_0^T \|(\Phi \tilde{Q}_M^{1/2})(t)\|_2^2 d\langle M \rangle_t] < \infty$.

The space of such integrands is a Hilbert space with respect to the scalar product $(\Phi_1, \Phi_2) \mapsto \mathbb{E}[\int_0^T \langle \Phi_1 \tilde{Q}_M^{1/2}, \Phi_2 \tilde{Q}_M^{1/2} \rangle d\langle M \rangle_t]$. Simple processes in $L(K)$ are examples of such integrands. Hence the closure of the set of simple

processes in this Hilbert space is itself a Hilbert subspace. We denote it as in [22] by $\Lambda^2(K; \mathcal{P}, M)$. More details and proofs can be found in [21] or [22].

In this paper we shall assume that there exists a measurable mapping $\mathcal{Q}(\cdot) : [0, T] \times \Omega \rightarrow L_1(K)$ such that $\mathcal{Q}(t)$ is symmetric, positive definite, $\langle\langle M \rangle\rangle_t = \int_0^t \mathcal{Q}(s) ds$, and $\mathcal{Q}(t) \leq \mathcal{Q}$ for some positive definite nuclear operator \mathcal{Q} on K . Thus $\tilde{\mathcal{Q}}_M(t) = \frac{\mathcal{Q}(t)}{q(t)}$ and $\langle M \rangle_t = \int_0^t q(s) ds$, with $q(t) = \text{tr}(\mathcal{Q}(t))$. Thus, if $\Phi \in \Lambda^2(K; \mathcal{P}, M)$,

$$\mathbb{E} \left[\left| \int_0^T \Phi(s) dM(s) \right|^2 \right] = \mathbb{E} \left[\int_0^T \|\Phi(s) \mathcal{Q}^{1/2}(s)\|_2^2 ds \right].$$

This equality will be used frequently in the proofs given in Section 5. The process $\mathcal{Q}(\cdot)$ will play an essential role in deriving the adjoint equation of the SPDE (1.1), as appearing in the equation (4.1) in Section 4; see in particular the discussion following equation (4.4).

3 Statement of the control problem

Let us consider the following space:

$$L_{\mathcal{F}}^2(0, T; E) := \{ \psi : [0, T] \times \Omega \rightarrow E, \text{ predictable and } \mathbb{E} \left[\int_0^T |\psi(t)|^2 dt \right] < \infty \},$$

where E is a separable Hilbert space. Suppose that \mathcal{O} is a separable Hilbert space with an inner product $\langle \cdot, \cdot \rangle_{\mathcal{O}}$, and U is a nonempty subset of \mathcal{O} . Denote by

$$\mathcal{U}_{ad} = \{ u(\cdot) : [0, T] \times \Omega \rightarrow \mathcal{O} \text{ s.t. } u(\cdot) \in L_{\mathcal{F}}^2(0, T; \mathcal{O}) \}.$$

This set is called the *set of admissible controls* and its elements are called *admissible controls*.

Now let us recall our SPDE:

$$\begin{cases} dx(t) = (A(t)x(t) + a(t, u(t))x(t) + b(t, u(t)))dt \\ \quad \quad \quad + [\langle \sigma(t, u(t)), x(t) \rangle_K + g(t, u(t))]dM(t), \\ x(0) = x_0 \in K, \end{cases} \quad (3.1)$$

and impose on it the following assumptions:

(i) $A(t, \omega)$ is a linear operator on K , \mathcal{P} -measurable, belongs to $L(V; V')$ uniformly in (t, ω) and satisfies the following two conditions.

- (1) $A(t, \omega)$ satisfies the coercivity condition:

$$2 \langle A(t, \omega) y, y \rangle + \alpha |y|_V^2 \leq \lambda |y|^2 \quad \text{a.e. } t \in [0, T], \quad \text{a.s. } \forall y \in V,$$

for some $\alpha, \lambda > 0$.

- (2) $\exists k_1 \geq 0$ such that for all (t, ω)

$$|A(t, \omega) y|_{V'} \leq k_1 |y|_V \quad \forall y \in V.$$

(ii) $a : \Omega \times [0, T] \times \mathcal{O} \rightarrow \mathbb{R}$, $b : \Omega \times [0, T] \times \mathcal{O} \rightarrow K$, $\sigma : \Omega \times [0, T] \times \mathcal{O} \rightarrow K$ and $g : \Omega \times [0, T] \times \mathcal{O} \rightarrow L_{\mathcal{Q}}(K)$ are predictable and bounded given mappings.

Definition 3.1 We say that $x = x^{u(\cdot)} \in L_{\mathcal{F}}^2(0, T; V)$ is a solution of (3.1) if $\forall \eta \in V$ (or any dense subset) and for almost all $(t, \omega) \in [0, T] \times \Omega$

$$\begin{aligned} \langle x(t), \eta \rangle_K &= \langle x_0, \eta \rangle_K + \int_0^t \langle A(s)x(s) + a(s, u(s))x(s) + b(s, u(s)), \eta \rangle_V ds \\ &\quad + \int_0^t \langle \eta, [\langle \sigma(s, u(s)), x(s) \rangle_K + g(s, u(s))] dM(s) \rangle_K. \end{aligned}$$

Given a bounded measurable mapping $\ell : [0, T] \times \mathcal{O} \rightarrow K$ and a fixed element G of K , we define the *cost functional* by:

$$J(u(\cdot)) := \mathbb{E} \left[\int_0^T \langle \ell(t, u(t)), x^{u(\cdot)}(t) \rangle_K dt + \langle G, x^{u(\cdot)}(T) \rangle_K \right], \quad u(\cdot) \in \mathcal{U}_{ad}. \quad (3.2)$$

It is easy to realize that under assumptions (i) and (ii) there exists a unique solution to (3.1) in $L_{\mathcal{F}}^2(0, T; K)$. This fact can be found in [13, Theorem 4.1, P. 105], [15, Theorem 2.10] or [4, Theorem 3.2], and also can be gleaned from [27]. Itô's formula for such SPDEs can be found in [14, Theorems 1, 2].

Our control problem is to minimize (3.2) over \mathcal{U}_{ad} . Any $u^*(\cdot) \in \mathcal{U}_{ad}$ satisfying

$$J(u^*(\cdot)) = \inf \{ J(u(\cdot)) : u(\cdot) \in \mathcal{U}_{ad} \} \quad (3.3)$$

is called an *optimal control*. The corresponding solution $x^{u^*(\cdot)}$ of (3.1), which we denote briefly by x^* and $(x^*, u^*(\cdot))$ are called respectively an *optimal solution* and an *optimal pair* of the stochastic optimal control problem (3.1)-(3.3).

The existence problem of optimal control can be developed from the works of [1], [2] and [29]. However, a special case can be found in [4].

4 Adjoint equation

Recall the SPDE (3.1) and the mappings in (3.2), and define the *Hamiltonian* $H : [0, T] \times \Omega \times K \times \mathcal{O} \times K \times L_2(K) \rightarrow \mathbb{R}$ for $(t, \omega, x, v, y, z) \in [0, T] \times \Omega \times K \times \mathcal{O} \times K \times L_2(K)$ by

$$\begin{aligned} H(t, \omega, x, v, y, z) &:= -\langle \ell(t, v), x \rangle_V - a(t, \omega, v) \langle x, y \rangle_K \\ &\quad - \langle b(t, \omega, v), y \rangle_K - \langle \tilde{\sigma}(t, \omega, x, v) \mathcal{Q}^{1/2}(t, \omega), z \rangle_2, \end{aligned} \quad (4.1)$$

where $\tilde{\sigma} : [0, T] \times \Omega \times K \times \mathcal{O} \rightarrow L_{\mathcal{Q}}(K)$ is defined by

$$\tilde{\sigma}(t, \omega, x, v) = \langle \sigma(t, \omega, v), x \rangle_K \Phi(x) + g(t, \omega, v)$$

with Φ being the constant mapping $\Phi : K \rightarrow L_{\mathcal{Q}}(K), x \mapsto \Phi(x) = \text{id}_K$. Then

$$\begin{aligned} \langle \tilde{\sigma}(t, \omega, x, v) \mathcal{Q}^{1/2}(t, \omega), z \rangle_2 &= \langle \langle \sigma(t, \omega, v), x \rangle_K (\Phi(x) + g(t, \omega, v)) \mathcal{Q}^{1/2}(t, \omega), z \rangle_2 \\ &= \langle \langle \mathcal{Q}^{1/2}(t, \omega), z \rangle_2 \sigma(t, \omega, v), x \rangle_K + \langle g(t, \omega, v) \mathcal{Q}^{1/2}(t, \omega), z \rangle_2 \\ &= \langle B(t, \omega, v) z, x \rangle_K + \langle g(t, \omega, v) \mathcal{Q}^{1/2}(t, \omega), z \rangle_2, \end{aligned}$$

where $B : [0, T] \times \Omega \times \mathcal{O} \rightarrow L(L_2(K), K)$ is defined such that

$$B(t, \omega, v) z = \langle \mathcal{Q}^{1/2}(t, \omega), z \rangle_2 \sigma(t, \omega, v). \quad (4.2)$$

Moreover,

$$\nabla_x H(t, \omega, x, v, y, z) = -\ell(t, v) - a(t, \omega, v) y - B(t, \omega, v) z. \quad (4.3)$$

The adjoint equation of (3.1) is the following BSPDE:

$$\begin{cases} dy^{u(\cdot)}(t) = - \left[A^*(t) y(t) - \nabla_x H(t, x^{u(\cdot)}(t), u(t), y^{u(\cdot)}(t), z^{u(\cdot)}(t) \mathcal{Q}^{1/2}(t)) \right] dt \\ \quad + z^{u(\cdot)}(t) dM(t) + dN^{u(\cdot)}(t), \quad 0 \leq t < T, \\ y^{u(\cdot)}(T) = G, \end{cases} \quad (4.4)$$

where $A^*(t)$ is the adjoint operator of $A(t)$.

It is important to realize that the presence of the process $\mathcal{Q}^{1/2}(\cdot)$ in the equation (4.4) is crucial in order for the mapping $\nabla_x H$ to be defined on the space $L_2(K)$, since the process $z^{u(\cdot)}$ need not be bounded as it is discussed in Section 2. This has to be taken always into account when dealing with BSPDEs and even BSDEs in infinite dimensions; cf. also [6].

The following theorem gives the solution to this BSPDE (4.4) in the sense that there exists a triple $(y^{u(\cdot)}, z^{u(\cdot)}, N^{u(\cdot)})$ in $L^2_{\mathcal{F}}(0, T; K) \times \Lambda^2(K; \mathcal{P}, M) \times \mathcal{M}^{2,c}_{[0,T]}(K)$ such that the following equality holds *a.s.* for all $t \in [0, T]$, $N(0) = 0$ and N is VSO to M :

$$\begin{aligned} y^{u(\cdot)}(t) = & \xi + \int_t^T \nabla_x H(s, x^{u(\cdot)}(s), u(s), y^{u(\cdot)}(s), z^{u(\cdot)}(s) \mathcal{Q}^{1/2}(s)) ds \\ & - \int_t^T z^{u(\cdot)}(s) dM(s) - \int_t^T dN^{u(\cdot)}(s). \end{aligned}$$

Theorem 4.1 *Assume that (i)–(ii) hold. Then there exists a unique solution $(y^{u(\cdot)}, z^{u(\cdot)}, N^{u(\cdot)})$ of the BSDE (4.4) in $L^2_{\mathcal{F}}(0, T; K) \times \Lambda^2(K; \mathcal{P}, M) \times \mathcal{M}^{2,c}_{[0,T]}(K)$.*

The proof of this theorem can be found in [3].

We shall denote briefly the solution of (4.4) corresponding to the optimal control $u^*(\cdot)$ by (y^*, z^*, N^*) .

5 Main results

In this section we shall derive and prove our main result on the maximum principle for optimal control of the SPDE (3.1) associated with cost functional (3.2) and value function (3.3) by using the results of the previous section on the adjoint equation (BSPDE). Before doing so, let us mention that the relationship between BSPDEs and maximum principle for some SPDEs is developed in several works, among them for instance are [24] and [30] and the references of Zhou cited therein. Other discussions in this respect can be found in [28] and [31] as well. Bensoussan in [11, Chapter 8] presents a stochastic maximum principle approach to the problem of stochastic control with partial information treating a general infinite dimensional setting and the adjoint equation is derived also there. Another work on the maximum principle that is connected to BSDEs can be found also in [7]. For an expanded discussion on the history of maximum principle we refer the reader to [30, P. 153–156]. And finally, one can find also useful information in Bensoussan’s lecture notes [9], [9] and Li & Yong [19] in addition to the references therein.

Our main theorem is the following.

Theorem 5.1 *Suppose (i)–(ii). If $(x^*, u^*(\cdot))$ is an optimal pair for the problem (3.1)–(3.3), then there exists a unique solution (y^*, z^*, N^*) to the corresponding BSEE (4.4) such that the following inequality holds:*

$$\begin{aligned} H(t, x^*(t), u, y^*(t), z^*(t) \mathcal{Q}^{1/2}(t)) \\ \leq H(t, x^*(t), u^*(t), y^*(t), z^*(t) \mathcal{Q}^{1/2}(t)) \end{aligned} \quad (5.1)$$

a.e. $t \in [0, T]$, a.s. $\forall u \in U$.

To start proving the theorem we need to develop some necessary estimates using the so-called spike variation method. For this we let $(x^*, u^*(\cdot))$ be the given optimal pair. Let $0 \leq t_0 < T$ be fixed such that $\mathbb{E} [|x(t_0)|_K^2] < \infty$ and $0 \leq \varepsilon < T - t_0$. Let u be a random variable taking its values in U , \mathcal{F}_{t_0} -measurable and $\sup_{\omega \in \Omega} |u(\omega)| < \infty$. Consider the following spike variation of the control $u^*(\cdot)$:

$$u_\varepsilon(t) = \begin{cases} u^*(t) & \text{if } t \in [0, T] \setminus [t_0, t_0 + \varepsilon] \\ u & \text{if } t \in [t_0, t_0 + \varepsilon]. \end{cases}$$

We can consider the $x^{u_\varepsilon(\cdot)}$ as the solution of the SPDE (3.1) corresponding to $u_\varepsilon(\cdot)$. We shall denote it briefly by x_ε . Note that $x_\varepsilon(t) = x^*(t)$ for all $0 \leq t \leq t_0$.

We shall divide the proof into several lemmas as follows.

Lemma 5.2 *Suppose (i)–(ii). Then*

$$\sup_{t_0 \leq t \leq t_0 + \varepsilon} \mathbb{E} [|x_\varepsilon(t)|_K^2] \leq C_1 (\mathbb{E} [|x^*(t_0)|_K^2] + C_2 \varepsilon) \quad (5.2)$$

for some positive constants C_1 and C_2 .

Proof. Observe first from (3.1) and (5.2) that, for $t_0 \leq t \leq t_0 + \varepsilon$,

$$\begin{aligned} x_\varepsilon(t) = x^*(t_0) + \int_{t_0}^t (A(s)x_\varepsilon(s) + a(s, u)x_\varepsilon(s) + b(s, u))ds \\ + \int_{t_0}^t [\langle \sigma(s, u), x_\varepsilon(s) \rangle_K + g(s, u)]dM(s). \end{aligned} \quad (5.3)$$

Therefore, by Itô's formula, assumption (i), Cauchy-Schwartz inequality and assumption (ii) we get

$$\begin{aligned}
 \mathbb{E} [|x_\varepsilon(t)|_K^2] + \alpha \mathbb{E} [\int_{t_0}^t |x_\varepsilon(s)|_V^2 ds] &\leq \mathbb{E} [|x^*(t_0)|_K^2] + \lambda \mathbb{E} [\int_{t_0}^t |x_\varepsilon(s)|_K^2 ds] \\
 &+ 2 \mathbb{E} [\int_{t_0}^t \langle a(s, u) x_\varepsilon(s), x_\varepsilon(s) \rangle_K ds] + 2 \mathbb{E} [\int_{t_0}^t \langle x_\varepsilon(s), b(s, u) \rangle_K ds] \\
 &+ 2 \mathbb{E} [\int_{t_0}^t \| \langle \sigma(s, u), x_\varepsilon(s) \rangle_K \text{id}_K \mathcal{Q}^{1/2}(s) \|_2^2 ds] + 2 \mathbb{E} [\int_{t_0}^t \| g(s, u) \mathcal{Q}^{1/2}(s) \|_2^2 ds] \\
 &\leq \mathbb{E} [|x^*(t_0)|_K^2] + \lambda \mathbb{E} [\int_{t_0}^t |x_\varepsilon(s)|_K^2 ds] + 2k_1 \mathbb{E} [\int_{t_0}^t |x_\varepsilon(s)|_K^2 ds] \\
 &\quad + k_2^2 \mathbb{E} [\int_{t_0}^t |x_\varepsilon(s)|_K^2 ds] + (t - t_0) \\
 &\quad + 2k_3^2 \| \mathcal{Q}^{1/2} \|_2^2 \mathbb{E} [\int_{t_0}^t |x_\varepsilon(s)|_K^2 ds] + 2k_4^2 \varepsilon \| \mathcal{Q}^{1/2} \|_2^2 \\
 &= (\lambda + 2k_1 + k_2^2 + 2k_3^2 \| \mathcal{Q}^{1/2} \|_2^2 (1 + \varepsilon)) \int_{t_0}^t \mathbb{E} [|x_\varepsilon(s)|_K^2] ds \\
 &\quad + (1 + 2k_4 k_4^2 \| \mathcal{Q}^{1/2} \|_2^2) \varepsilon + \mathbb{E} [|x^*(t_0)|_K^2]. \quad (5.4)
 \end{aligned}$$

In the last part of this inequality we have used the boundedness in assumption (ii) of the mappings a, b, σ, g respectively to get the constants $k_1 - k_4$.

Thus, in particular, by applying Gronwall's inequality to (5.4) we obtain (5.2) with

$$C_1 = e^\varepsilon (\lambda + 2k_1 + k_2^2 + 2k_3^2 \| \mathcal{Q}^{1/2} \|_2^2 (1 + \varepsilon))$$

and

$$C_2 = 1 + 2k_4^2 \| \mathcal{Q}^{1/2} \|_2^2.$$

This completes the proof. ■

Lemma 5.3 *Suppose (i)–(ii). Then*

$$\sup_{t_0 + \varepsilon \leq t \leq T} \mathbb{E} [|x_\varepsilon(t)|_K^2] \leq C_3 (\mathbb{E} [|x^*(t_0)|_K^2] + C_4 \varepsilon + 1) \quad (5.5)$$

for some positive constants C_3 and C_4 .

Proof. For $t_0 + \varepsilon \leq t \leq T$, it follows that

$$\begin{aligned} x_\varepsilon(t) &= x^*(t_0 + \varepsilon) + \int_{t_0 + \varepsilon}^t (A(s)x_\varepsilon(s) + a(s, u^*(s))x_\varepsilon(s) + b(s, u^*(s)))ds \\ &\quad + \int_{t_0 + \varepsilon}^t [\langle \sigma(s, u^*(s)), x_\varepsilon(s) \rangle_K + g(s, u^*(s))]dM(s). \end{aligned} \quad (5.6)$$

Thus mimicking the proof of Lemma 5.2 and then applying inequality (5.2) easily yields (5.5). ■

Lemma 5.4 Suppose (i)–(ii). Let $\xi_\varepsilon(t) = x_\varepsilon(t) - x^*(t)$, for $t \in [0, T]$. Then

$$\sup_{t_0 + \varepsilon \leq t \leq T} \mathbb{E} [|\xi_\varepsilon(t)|_K^2] = O(\varepsilon). \quad (5.7)$$

Proof. It is easy to get for $t \in [t_0 + \varepsilon, T]$,

$$\begin{aligned} \xi_\varepsilon(t) &= \xi_\varepsilon(t_0 + \varepsilon) + \int_{t_0 + \varepsilon}^t (A(s)\xi_\varepsilon(s) + a(s, u^*(s))\xi_\varepsilon(s))ds \\ &\quad + \int_{t_0 + \varepsilon}^t \langle \sigma(s, u^*(s)), \xi_\varepsilon(s) \rangle_K dM(s). \end{aligned} \quad (5.8)$$

Hence, as done in the proof of Lemma 5.2, we get

$$\sup_{t_0 + \varepsilon \leq t \leq T} \mathbb{E} [|\xi_\varepsilon(t)|_K^2] \leq C_5 \mathbb{E} [|\xi_\varepsilon(t_0 + \varepsilon)|_K^2]. \quad (5.9)$$

On the other hand, for $t_0 \leq t \leq t_0 + \varepsilon$ we have $\xi_\varepsilon(t_0) = 0$ and

$$\begin{aligned} \xi_\varepsilon(t) &= \int_{t_0}^t \left[A(s)\xi_\varepsilon(s) + (a(s, u) - a(s, u^*(s)))x_\varepsilon(s) \right. \\ &\quad \left. + (b(s, u) - b(s, u^*(s))) + a(s, u^*(s))\xi_\varepsilon(s) \right] ds \\ &\quad + \int_{t_0}^t \left[\langle \sigma(s, u) - \sigma(s, u^*(s)), x_\varepsilon(s) \rangle_K \right. \\ &\quad \left. + (g(s, u) - g(s, u^*(s))) + \langle \sigma(s, u^*(s)), \xi_\varepsilon(s) \rangle_K \right] dM(s). \end{aligned} \quad (5.10)$$

Hence by Itô's formula, assumption (i), Cauchy-Schwartz inequality and assumption (ii) it follows that

$$\begin{aligned}
& \mathbb{E} [| \xi_\varepsilon(t) |_K^2] + \alpha \mathbb{E} [\int_{t_0}^t | \xi_\varepsilon(s) |_V^2 ds] \\
& \leq \lambda \mathbb{E} [\int_{t_0}^t | \xi_\varepsilon(s) |_K^2 ds] + 2 \mathbb{E} [\int_{t_0}^t \langle \xi_\varepsilon(s), (a(s, u) - a(s, u^*(s))) x_\varepsilon(s) \rangle_K ds] \\
& \quad + 2 \mathbb{E} [\int_{t_0}^t \langle \xi_\varepsilon(s), b(s, u) - b(s, u^*(s)) \rangle_K ds] \\
& \quad + 2 \mathbb{E} [\int_{t_0}^t \langle \xi_\varepsilon(s), a(s, u^*(s)) \xi_\varepsilon(s) \rangle_K ds] \\
& \quad + 3 \mathbb{E} [\int_{t_0}^t || \langle \sigma(s, u) - \sigma(s, u^*(s)), x_\varepsilon(s) \rangle_K \text{id}_K \mathcal{Q}^{1/2}(s) ||_2^2 ds] \\
& \quad + 3 \mathbb{E} [\int_{t_0}^t || \langle \sigma(s, u^*(s)), \xi_\varepsilon(s) \rangle_K \text{id}_K \mathcal{Q}^{1/2}(s) ||_2^2 ds] \\
& \quad + 3 \mathbb{E} [\int_{t_0}^t || (g(s, u) - g(s, u^*(s))) \mathcal{Q}^{1/2}(s) ||_2^2 ds] \\
& \leq (\lambda + 4k_1^2 + k_2^2 + 2k_1 + 3k_3^2 \cdot ||\mathcal{Q}^{1/2}||_2^2) \int_{t_0}^t \mathbb{E} [| \xi_\varepsilon(s) |_K^2] ds \\
& \quad + (6k_3^2 ||\mathcal{Q}^{1/2}||_2^2 + 1) \int_{t_0}^{t_0+\varepsilon} \mathbb{E} [| x_\varepsilon(s) |_K^2] ds + (1 + 12k_4^2)\varepsilon \\
& \leq (\lambda + 4k_1^2 + k_2^2 + 2k_1 + 3k_3^2 ||\mathcal{Q}^{1/2}||_2^2) \int_{t_0}^t \mathbb{E} [| \xi_\varepsilon(s) |_K^2] ds \\
& \quad + (6k_3^2 ||\mathcal{Q}^{1/2}||_2^2 + 1) C_1 \cdot (\mathbb{E} [| x^*(t_0) |_K^2] + C_2 \varepsilon) \varepsilon + (1 + 12k_4^2)\varepsilon. \tag{5.11}
\end{aligned}$$

Therefore Gronwall's inequality gives

$$\sup_{t_0 \leq t \leq t_0+\varepsilon} \mathbb{E} [| \xi_\varepsilon(t) |_K^2] \leq C_6(\varepsilon) \cdot \varepsilon, \tag{5.12}$$

where

$$\begin{aligned}
C_6(\varepsilon) = e^{(\lambda+4k_1^2+k_2^2+2k_1+3k_3^2||\mathcal{Q}^{1/2}||_2^2)\varepsilon} \cdot \Big[& (6k_3^2 ||\mathcal{Q}^{1/2}||_2^2 + 1) C_1 \cdot (\mathbb{E} [| x^*(t_0) |_K^2] \\
& + C_2 \varepsilon) + 1 + 12k_4^2 \Big].
\end{aligned}$$

Now by applying (5.12) in (5.9) it yields eventually

$$\sup_{t_0+\varepsilon \leq t \leq T} \mathbb{E} [| \xi_\varepsilon(t) |_K^2] \leq C_5 C_6(\varepsilon) \cdot \varepsilon. \tag{5.13}$$

Thus (5.7) follows. ■

In the following result we shall try to compute $\mathbb{E}[\langle y^*(t_0 + \varepsilon), \xi(t_0 + \varepsilon) \rangle_K]$.

Lemma 5.5 *Suppose (i)–(ii). We have*

$$\begin{aligned}
& \mathbb{E} \left[\langle y^*(t_0 + \varepsilon), \xi_\varepsilon(t_0 + \varepsilon) \rangle_K + \int_{t_0}^{t_0 + \varepsilon} \langle \ell(t, u^*(t)), \xi_\varepsilon(t) \rangle_K dt \right] \\
&= \mathbb{E} \left[\int_{t_0}^{t_0 + \varepsilon} \langle y^*(t), (a(t, u) - a(t, u^*(t))) x_\varepsilon(t) \rangle_K dt \right] \\
&+ \mathbb{E} \left[\int_{t_0}^{t_0 + \varepsilon} \langle y^*(t), b(t, u) - b(t, u^*(t)) \rangle_K dt \right] \\
&+ \mathbb{E} \left[\int_{t_0}^{t_0 + \varepsilon} \langle \sigma(t, u) - \sigma(t, u^*(t)), x_\varepsilon(t) \rangle_K \langle \mathcal{Q}^{1/2}(t), z^*(t) \mathcal{Q}^{1/2}(t) \rangle_2 dt \right] \\
&+ \mathbb{E} \left[\int_{t_0}^{t_0 + \varepsilon} \langle (g(t, u) - g(t, u^*(t))) \mathcal{Q}^{1/2}(t), z^*(t) \mathcal{Q}^{1/2}(t) \rangle_2 dt \right] \quad (5.14)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[\langle y^*(t_0 + \varepsilon), \xi_\varepsilon(t_0 + \varepsilon) \rangle_K] &= \mathbb{E} \left[\int_{t_0 + \varepsilon}^T \langle \ell(t, u^*(t)), \xi_\varepsilon(t) \rangle_K dt \right] \\
&+ \mathbb{E}[\langle G, \xi_\varepsilon(T) \rangle_K]. \quad (5.15)
\end{aligned}$$

Proof. Note that for $t_0 \leq t \leq t_0 + \varepsilon$ we have $\xi_\varepsilon(t_0) = 0$ and (5.10). Therefore by using Itô's formula to (5.10) together with (4.4), (4.3) and (4.2) we get easily (5.14). The equality in (5.15) is proved similarly with the help of (5.8). ■

Lemma 5.6 *Suppose (i)–(ii). We have*

$$\begin{aligned}
0 &\leq \mathbb{E} \left[\int_{t_0}^{t_0 + \varepsilon} \langle \ell(t, u) - \ell(t, u^*(t)), x^*(t) \rangle_K dt \right] \\
&+ \mathbb{E} \left[\int_{t_0}^{t_0 + \varepsilon} \langle y^*(t), (a(t, u) - a(t, u^*(t))) x^*(t) \rangle_K dt \right] \\
&+ \mathbb{E} \left[\int_{t_0}^{t_0 + \varepsilon} \langle \sigma(t, u) - \sigma(t, u^*(t)), x^*(t) \rangle_K \langle \mathcal{Q}^{1/2}(t), z^*(t) \mathcal{Q}^{1/2}(t) \rangle_2 dt \right] \\
&+ \mathbb{E} \left[\int_{t_0}^{t_0 + \varepsilon} \langle y^*(t), b(t, u) - b(t, u^*(t)) \rangle_K dt \right] \\
&+ \mathbb{E} \left[\int_{t_0}^{t_0 + \varepsilon} \langle (g(t, u) - g(t, u^*(t))) \mathcal{Q}^{1/2}(t), z^*(t) \mathcal{Q}^{1/2}(t) \rangle_2 dt \right] + o(\varepsilon). \quad (5.16)
\end{aligned}$$

Proof. Since $u^*(\cdot)$ is optimal, we have

$$\begin{aligned}
0 &\leq J(u_\varepsilon(\cdot)) - J(u^*(\cdot)) \\
&= \mathbb{E} \left[\int_0^T (\langle \ell(t, u_\varepsilon(t)), x_\varepsilon(t) \rangle_K - \langle \ell(t, u^*(t)), x^*(t) \rangle_K) dt \right] \\
&\quad + \mathbb{E} [\langle G, x_\varepsilon(T) \rangle_K - \langle G, x^*(T) \rangle_K] \\
&= \mathbb{E} \left[\int_{t_0}^{t_0+\varepsilon} (\langle \ell(t, u) - \ell(t, u^*(t)), x_\varepsilon(t) \rangle_K + \langle \ell(t, u^*(t)), \xi_\varepsilon(t) \rangle_K) dt \right] \\
&\quad + \mathbb{E} \left[\int_{t_0+\varepsilon}^T \langle \ell(t, u^*(t)), \xi_\varepsilon(t) \rangle_K dt + \langle G, \xi_\varepsilon(T) \rangle_K \right].
\end{aligned}$$

Hence using Lemma 5.5 (5.15) in this inequality gives

$$\begin{aligned}
0 \leq \mathbb{E} \left[\int_{t_0}^{t_0+\varepsilon} (\langle \ell(t, u) - \ell(t, u^*(t)), x_\varepsilon(t) \rangle_K dt + \langle \ell(t, u^*(t)), \xi_\varepsilon(t) \rangle_K) dt \right] \\
+ \mathbb{E} [\langle y^*(t_0 + \varepsilon), \xi_\varepsilon(t_0 + \varepsilon) \rangle_K]. \quad (5.17)
\end{aligned}$$

Again by Lemma 5.5 (5.14) inequality (5.17) becomes

$$\begin{aligned}
0 \leq \mathbb{E} \left[\int_{t_0}^{t_0+\varepsilon} \langle \ell(t, u) - \ell(t, u^*(t)), x_\varepsilon(t) \rangle_K dt \right] \\
+ \mathbb{E} \left[\int_{t_0}^{t_0+\varepsilon} \langle y^*(t), (a(t, u) - a(t, u^*(t))) x_\varepsilon(t) \rangle_K dt \right] \\
+ \mathbb{E} \left[\int_{t_0}^{t_0+\varepsilon} \langle y^*(t), b(t, u) - b(t, u^*(t)) \rangle_K dt \right] \\
+ \mathbb{E} \left[\int_{t_0}^{t_0+\varepsilon} \langle \sigma(t, u) - \sigma(t, u^*(t)), x_\varepsilon(t) \rangle_K \langle \mathcal{Q}^{1/2}(t), z^*(t) \mathcal{Q}^{1/2}(t) \rangle_2 dt \right] \\
+ \mathbb{E} \left[\int_{t_0}^{t_0+\varepsilon} \langle (g(t, u) - g(t, u^*(t))) \mathcal{Q}^{1/2}(t), z^*(t) \mathcal{Q}^{1/2}(t) \rangle_2 dt \right]. \quad (5.18)
\end{aligned}$$

On the other hand, assumption (ii) and Lemma 5.4 imply

$$\begin{aligned}
& \frac{1}{\varepsilon} \mathbb{E} \left[\int_{t_0}^{t_0+\varepsilon} \langle y^*(t), (a(t, u) - a(t, u^*(t))) \xi_\varepsilon(t) \rangle_K dt \right] \\
& \leq C_7 \left(\frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \mathbb{E} \left(|y^*(t)|_K \cdot |\xi_\varepsilon(t)|_K \right) dt \right) \\
& \leq C_7 \left(\frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \left(\left(\frac{\varepsilon^{1/3}}{2} \right) \mathbb{E} [|y^*(t)|_K^2] + \left(\frac{1}{2\varepsilon^{1/3}} \right) \mathbb{E} [|\xi_\varepsilon(t)|_K^2] \right) dt \right) \\
& \leq C_8 \left(\varepsilon^{1/3} \left(\frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \mathbb{E} [|y^*(t)|_K^2] dt + \left(\frac{1}{\varepsilon} \right) \varepsilon \left(\frac{1}{\varepsilon^{1/3}} \right) \varepsilon \right) \right) \rightarrow 0, \quad (5.19)
\end{aligned}$$

as $\varepsilon \rightarrow 0$, provided that t_0 is a Lebesgue point of the function $t \mapsto \mathbb{E} [|y^*(t)|_K^2]$, for some positive constants C_7 and C_8 .

Similarly,

$$\begin{aligned}
& \frac{1}{\varepsilon} \mathbb{E} \left[\int_{t_0}^{t_0+\varepsilon} \left(\langle \ell(t, u) - \ell(t, u^*(t)), \xi_\varepsilon(t) \rangle_K \right. \right. \\
& \quad \left. \left. + \langle \sigma(t, u) - \sigma(t, u^*(t)), \xi_\varepsilon(t) \rangle_K \langle \mathcal{Q}^{1/2}(t), z^*(t) \mathcal{Q}^{1/2}(t) \rangle_2 \right) dt \right] \rightarrow 0, \quad (5.20)
\end{aligned}$$

as $\varepsilon \rightarrow 0$, provided that t_0 is a Lebesgue point of the function $t \mapsto \mathbb{E} [|z^*(t) \mathcal{Q}^{1/2}(t)|_2^2]$.

Therefore, by applying (5.19) and (5.20) in (5.18) we obtain (5.16). ■

We are now ready to complete the proof of Theorem 5.1.

Proof of Theorem 5.1 Divide (5.16) in Lemma 5.6 by ε and let $\varepsilon \rightarrow 0$ to get

$$\begin{aligned}
& \mathbb{E} \left[\langle \ell(t_0, u) - \ell(t_0, u^*(t_0)), x^*(t_0) \rangle_K + \langle y^*(t_0), (a(t_0, u) - a(t_0, u^*(t_0))) x^*(t_0) \rangle_K \right] \\
& \quad + \mathbb{E} \left[\langle y^*(t_0), b(t_0, u) - b(t_0, u^*(t_0)) \rangle_K \right] \\
& \quad + \mathbb{E} \left[\langle \sigma(t_0, u) - \sigma(t_0, u^*(t_0)), x^*(t_0) \rangle_K \langle \mathcal{Q}^{1/2}(t_0), z^*(t_0) \mathcal{Q}^{1/2}(t_0) \rangle_2 \right] \\
& \quad + \mathbb{E} \left[\langle (g(t_0, u) - g(t_0, u^*(t_0))) \mathcal{Q}^{1/2}(t_0), z^*(t_0) \mathcal{Q}^{1/2}(t_0) \rangle_2 \right] \geq 0.
\end{aligned}$$

Consequently,

$$\begin{aligned}
& H(t_0, x^*(t_0), u, y^*(t_0), z^*(t_0) \mathcal{Q}^{1/2}(t_0)) \\
& \leq H(t_0, x^*(t_0), u^*(t_0), y^*(t_0), z^*(t_0) \mathcal{Q}^{1/2}(t_0)).
\end{aligned}$$

Hence (5.1) holds by a standard argument as for example in [30, Chapet 3], and the proof of Theorem 5.1 is then complete.

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